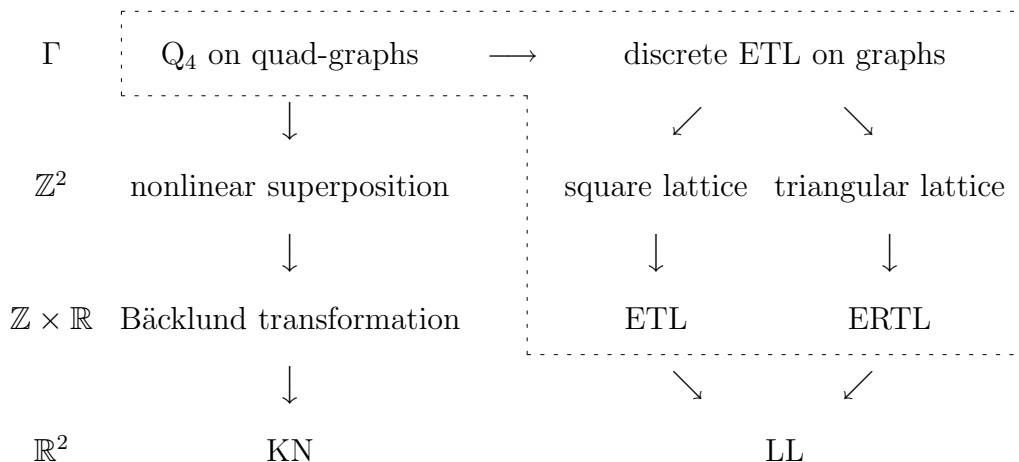


Q₄

V.E. ADLER^{*,a} AND YU.B. SURIS^{*,s}

1 Introduction

One of the most fascinating and technically demanding parts of the theory of two-dimensional integrable systems constitute the models with the spectral parameter on an elliptic curve. These include Landau-Lifshitz (LL) and Krichever-Novikov (KN) equations, as well as elliptic Toda lattice (ETL) and elliptic Ruijsenaars-Toda lattice (ERTL). The latter two systems are less known; they will be discussed in detail in this paper (the word “elliptic” in their names refers to their relation to an elliptic curve). We will explain how all these models can be unified on the basis of a single equation introduced in [2] as the nonlinear superposition formula for Bäcklund transformations of the KN equation, and studied further in [6, 18]. We will denote it Q_4 , following [6]. This discrete model plays the role of the “master equation” in the (sl_2 part of) 2D integrability: most of other models in this area can be obtained from this one by certain limiting procedures. Some of the interrelations have been known for a while, but several important parts of the picture were discovered only recently. The tower of continualizations is illustrated by the following diagram:



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Here the left column denotes the independent variables domain of the system. The first line contains totally discrete systems on planar graphs studied recently in [3, 5, 6, 7, 9, 8], see also [15, 19]. The link between Q_4 and discrete ETL is a sort of reduction related with black-white coloring of quad-graphs, which was discovered in [7]. The transition $\Gamma \rightarrow \mathbb{Z}^2$ is just the specification of the general construction to the most important cases of regular lattices. Discrete equations in this more traditional setting were studied for a long time, see e.g. [11, 20, 17, 23, 24, 25, 3, 4].

The arrows $\mathbb{Z}^2 \rightarrow \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}^2$ have a double meaning: a continuous system can be considered as a (non-classical) symmetry of the discrete one, or alternatively it can be obtained by a continuous limit. The amount of papers grows exponentially at each step down the diagram. We mention only the classical sources [26, 10, 21] and classification results on lattice systems due to the Siberian school [27, 22, 28, 1].

The contents of our paper correspond to the route along the arrows in the framed part of the above diagram (the rest part can be found, e.g. in [6, 4]). In the main text we discuss only the non-degenerate case related to an elliptic curve. Several more simple cases can be obtained without additional efforts by some specification of the general formulae, for example by substituting $\sinh(x)$ or x instead of the Weierstrass function $\sigma(x)$. This corresponds to a degeneration of an elliptic curve into a rational one. All constructions of the present paper remain valid under such degenerations, which constitute the list Q_1 – Q_4 of integrable discrete equations on quad-graphs placed in Appendix A. Actually, it contains six equations, since the parameter δ in equations Q_1 and Q_3 can be scaled to either 0 or 1. This list represents the most essential part of the classification result obtained in [6].

2 Discrete equations on quad-graphs

We start with equations

$$Q(u_0, u_1, u_2, u_3; \alpha, \beta) = 0 \quad (1)$$

on quad-graphs (that is, planar graphs with quadrilateral faces). Here the “fields” $u_k \in \mathbb{CP}^1$ are assigned cyclically to the vertices of any face of the graph and the parameters $\alpha, \beta \in \mathbb{C}$ are assigned to its edges. It is required that opposite edges of any face carry one and the same parameter (see Fig. 1). According to [7, 18], *integrability* of such equation is synonymous with *3D consistency*, see below. The most important representative of integrable equations (1) is the equation Q_4 , which we give here in the nice form derived in [18]:

$$\begin{aligned} & A((u_0 - b)(u_3 - b) - (a - b)(c - b))((u_1 - b)(u_2 - b) - (a - b)(c - b)) \\ & + B((u_0 - a)(u_1 - a) - (b - a)(c - a))((u_3 - a)(u_2 - a) - (b - a)(c - a)) \\ & = ABC(a - b), \end{aligned} \quad (Q_4)$$

where the points

$$(a, A) = (\wp(\alpha), \wp'(\alpha)), \quad (b, B) = (\wp(\beta), \wp'(\beta)), \quad (c, C) = (\wp(\beta - \alpha), \wp'(\beta - \alpha))$$

belong to the elliptic curve $\mathcal{E} = \{A^2 = r(a)\}$, and $r(a) = 4a^3 - g_2a - g_3$ is the notation for the Weierstrass polynomial used throughout the paper. So, in this case it is natural

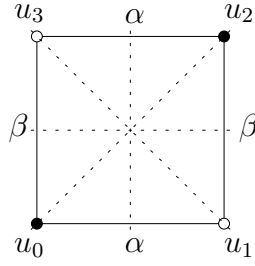


Fig. 1: The basic quadrilateral and D_4 group of symmetry

to assume that the parameters α, β belong to the period parallelogram of \mathcal{E} rather than to \mathbb{C} . A collection of formulas related to \mathcal{E} and to the corresponding Weierstrass elliptic functions is put in Appendix B.

It is obvious that Q_4 is invariant under reflections in the diagonals: $u_0 \leftrightarrow u_2$, $\alpha \leftrightarrow \beta$ and $u_1 \leftrightarrow u_3$, $\alpha \leftrightarrow \beta$, as well as under reflections $(u_0, u_3) \leftrightarrow (u_1, u_2)$ and $(u_0, u_1) \leftrightarrow (u_3, u_2)$, so that this equation admits the symmetry group D_4 of the square.

Actually, the symmetry of Q_4 is even more rich. This will follow from another form of this equation, which is of a fundamental importance also for many other reasons, as will be shown below.

Proposition 1. *Equation Q_4 is equivalent, under the point transformations $u_i = \wp(x_i)$, to equation*

$$F(x_0, x_1; \alpha)/F(x_0, x_3; \beta) = F(x_0, x_2; \alpha - \beta), \quad (2)$$

where

$$F(x_0, x_1; \alpha) = \frac{\sigma(x_0 + x_1 + \alpha)\sigma(x_0 - x_1 + \alpha)}{\sigma(x_0 + x_1 - \alpha)\sigma(x_0 - x_1 - \alpha)}. \quad (3)$$

This proposition was proved in [6]. Equation (2) is called the *multiplicative three-leg form* of (1) *centered at x_0* . Sometimes it is more convenient to use the *additive three-leg form*

$$f(x_0, x_1; \alpha) - f(x_0, x_3; \beta) = f(x_0, x_2; \alpha - \beta), \quad (4)$$

where

$$f(x_0, x_1; \alpha) = \frac{1}{2} \log F(x_0, x_1; \alpha). \quad (5)$$

The functions f for equations Q_1 – Q_3 , as well as the corresponding point transformations of the fields and parameters are listed in Appendix A.

Proposition 1 is checked with the help of some identities for elliptic functions listed in Appendix B. With the help of (B.4) one sees immediately that the function (3) is fractional-linear in $u_1 = \wp(x_1)$ (but not in $u_0 = \wp(x_0)$):

$$F(x_0, x_1; \alpha) = \frac{\wp(x_1) - \wp(x_0 + \alpha)}{\wp(x_1) - \wp(x_0 - \alpha)} \cdot \frac{\sigma^2(x_0 + \alpha)}{\sigma^2(x_0 - \alpha)}. \quad (6)$$

Therefore, the formula (2) yields an equation which is already affine-linear in u_1, u_2, u_3 , but with a complicated dependence on x_0 . Some additional transformations based on (B.1), (B.3) are needed in order to show that this equation coincides with Q_4 up to some

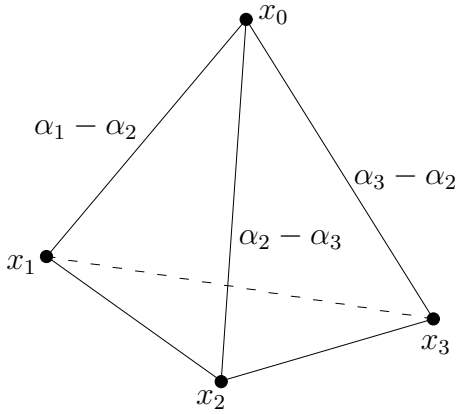


Fig. 2: S_4 symmetry of discrete KN

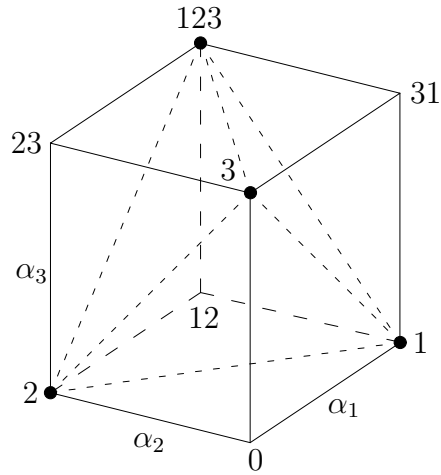


Fig. 3: 3D consistency

factor depending on x_0 . Note that eq. (2) defines, strictly speaking, a correspondence, since each x_i is defined up to a sign and up to shifts by periods of the function $\wp(x)$.

The function F has an obvious symmetry property

$$F(x_0, x_1; \alpha) = \frac{1}{F(x_0, x_1; -\alpha)} \quad (7)$$

which implies one of the D_4 symmetries $x_1 \leftrightarrow x_3$, $\alpha \leftrightarrow \beta$. However, another symmetry $(x_0, x_3) \leftrightarrow (x_1, x_2)$ of the equation Q_4 becomes hidden in its three-leg form (2) centered at x_0 . *Due to this symmetry the three-leg form can be centered at an arbitrary vertex.*

On the other hand, three-leg form exhibits one more symmetry which remains hidden in the rational form Q_4 , namely the fact that the diagonals of the quadrilateral are on the equal footing with its edges. Indeed, denote $\alpha = \alpha_1 - \alpha_2$, $\beta = \alpha_3 - \alpha_2$, then eq. (2) takes the form

$$F(x_0, x_1; \alpha_1 - \alpha_2)F(x_0, x_2; \alpha_2 - \alpha_3)F(x_0, x_3; \alpha_3 - \alpha_1) = 1. \quad (8)$$

This means that the full symmetry group of discrete KN equation is not the D_4 group of the square on Fig. 1, but the symmetric group S_4 of the tetrahedron on Fig. 2. Probably, there does not exist the form of equation, in which the whole group becomes apparent. Indeed, even in the simplest case of equation $(Q_1)_{\delta=0}$ (see Appendix A) which reads $CrossRatio(x_0, x_1, x_2, x_3) = \alpha/\beta$, this full group S_4 is encoded in the transformation properties of the cross-ratio.

As the first application of the three-leg form, we will prove that it (together with the above mentioned symmetry properties) implies the fundamental property of 3D consistency, which can be accepted as the definition of integrability for equations of the form (1). Recall [7, 18] that equation (1) is called 3D-consistent, if the following holds. For arbitrary initial data u_0, u_1, u_2, u_3 , define the values u_{12}, u_{23}, u_{31} (the enumeration, which is different from the one used elsewhere in this paper, is explained on Fig. 3) according

to the equations

$$Q(u_0, u_1, u_{12}, u_2; \alpha_1, \alpha_2) = 0, \quad (9)$$

$$Q(u_0, u_2, u_{23}, u_3; \alpha_2, \alpha_3) = 0, \quad (10)$$

$$Q(u_0, u_3, u_{31}, u_1; \alpha_3, \alpha_1) = 0, \quad (11)$$

which correspond to three faces of the cube adjacent to the vertex 0. Then equations

$$Q(u_1, u_{12}, u_{123}, u_{31}; \alpha_2, \alpha_3) = 0, \quad (12)$$

$$Q(u_2, u_{23}, u_{123}, u_{12}; \alpha_3, \alpha_1) = 0, \quad (13)$$

$$Q(u_3, u_{31}, u_{123}, u_{23}; \alpha_1, \alpha_2) = 0, \quad (14)$$

corresponding to the rest faces, define one and the same value u_{123} .

Proposition 2. *Suppose that equation (1) admits the D_4 symmetry group and is equivalent, under some point transformation $u = \phi(x)$, $a = \rho(\alpha)$, to the three-leg equation (2). Then it is 3D-consistent.*

Proof. It is enough to show that if u_{123} is defined by equation (12), then (13) is fulfilled as well. Rewrite equations (9), (11), (12) (corresponding to the faces adjacent to the vertex 1) in the three-leg forms centered at x_1 (recall that this is possible due to the symmetry properties):

$$\begin{aligned} F(x_1, x_{31}; \alpha_3) / F(x_1, x_0; \alpha_1) &= F(x_1, x_3; \alpha_3 - \alpha_1), \\ F(x_1, x_{12}; \alpha_2) / F(x_1, x_0; \alpha_1) &= F(x_1, x_2; \alpha_2 - \alpha_1), \\ F(x_1, x_{12}; \alpha_2) / F(x_1, x_{31}; \alpha_3) &= F(x_1, x_{123}; \alpha_2 - \alpha_3). \end{aligned}$$

From these there follows the equation relating the fields at the vertices of the dashed tetrahedron on Fig. 3:

$$F(x_1, x_2; \alpha_2 - \alpha_1) / F(x_1, x_3; \alpha_3 - \alpha_1) = F(x_1, x_{123}; \alpha_2 - \alpha_3).$$

This is nothing but the three-leg form of the equation

$$Q(u_1, u_2, u_3, u_{123}; \alpha_2 - \alpha_1, \alpha_2 - \alpha_3) = 0, \quad (15)$$

centered at x_1 . This can be centered at x_2 , as well, resulting in the cyclic shift of indices:

$$F(x_2, x_3; \alpha_3 - \alpha_2) / F(x_2, x_1; \alpha_1 - \alpha_2) = F(x_2, x_{123}; \alpha_3 - \alpha_1).$$

The latter equation, together with the three-leg forms of equations (9), (10) centered at x_2 , leads to the three-leg form of (13), as required. \square

3 Discrete Toda systems on graphs

Existence of the three-leg form allows us to establish a direct link to discrete Toda systems on graphs introduced in [5]. This link was discovered in [7] and is described as follows.

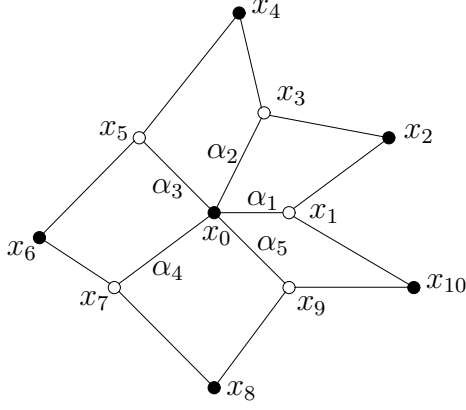


Fig. 4: Faces adjacent to the vertex x_0 .

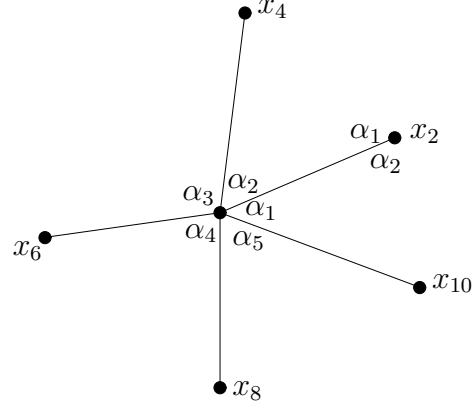


Fig. 5: The black graph star of the vertex x_0 .

Suppose that the quad-graph Γ is bi-partite, i.e. its vertices can be colored black and white so that the ends of any edge are of different colors. Consider the graph Γ_0 whose set of vertices $V(\Gamma_0)$ consists of the black vertices of Γ , and whose edges are diagonals of quadrilateral faces of Γ . Consider a vertex $x_0 \in V(\Gamma_0)$ (for brevity we will identify vertices with the field variables sitting there), and let x_0 be a common vertex of n adjacent quadrilateral faces $(x_0, x_{2j-1}, x_{2j}, x_{2j+1}) \in F(\Gamma)$, $j = 1, \dots, n$, like shown on Fig. 4. Then, multiplying the equations of the type (2) centered in x_0 for all these faces, we see that the white vertices x_{2j-1} cancel out and the black vertices x_{2j} satisfy the equation on the star of the vertex x_0 :

$$\prod_{j=1}^n F(x_0, x_{2j}; \alpha_j - \alpha_{j+1}) = 1, \quad (16)$$

or, in the additive form,

$$\sum_{j=1}^n f(x_0, x_{2j}; \alpha_j - \alpha_{j+1}) = 0. \quad (17)$$

This is a *discrete Toda system on the graph Γ_0* . (Of course, a similar Toda type system holds also for the white graph Γ_1 which is dual to Γ_0 .) The parameters α_j which were assigned originally to the edges $(x_0, x_{2j-1}) \in E(\Gamma)$, label now the corners of the faces of Γ_0 . Notice that only the *differences* of these parameters appear in equations (17), so that simultaneous shifting all the parameters of the equations on the original quad-graph $\alpha \rightarrow \alpha + \lambda$ leads to the same Toda system on Γ_0 . This shift plays the role of the spectral parameter in the zero curvature representation of the Toda system.

As established in [7, 18], any 3D consistent equation (1) is its own zero curvature representation. We will show that it plays this role not only for itself, but for a whole set of related systems, including discrete Toda systems. For this aim, rewrite eq. Q₄ on the quadrilateral $(u_0, u_1, u_2, u_3) \in F(\Gamma)$ as

$$\begin{aligned} k_0 u_0 u_1 u_2 u_3 + k_1 (u_1 u_2 u_3 + u_0 u_2 u_3 + u_0 u_1 u_3 + u_0 u_1 u_2) + k_2 (u_0 u_2 + u_1 u_3) \\ + k_3 (u_0 u_1 + u_2 u_3) + k_4 (u_0 u_3 + u_1 u_2) + k_5 (u_0 + u_1 + u_2 + u_3) + k_6 = 0. \end{aligned}$$

(Concrete expressions of the coefficients k_i in terms of (a, A) and (b, B) can be found in the original paper [2].) The field u_3 is a fractional-linear function (Möbius transformation) of u_1 with coefficients depending on u_0, u_2 and α_1, α_2 . Denote the matrix of this Möbius transformation as

$$U(u_0, u_2; \alpha_1, \alpha_2) = \frac{1}{D} \begin{pmatrix} -k_1 u_0 u_2 - k_3 u_0 - k_4 u_2 - k_5 & -k_2 u_0 u_2 - k_5(u_0 + u_2) - k_6 \\ k_0 u_0 u_2 + k_1(u_0 + u_2) + k_2 & k_1 u_0 u_2 + k_4 u_0 + k_3 u_2 + k_5 \end{pmatrix}, \quad (18)$$

where D is the normalizing factor such that $\det U = 1$. Considering now n quadrilateral faces $(u_0, u_{2j-1}, u_{2j}, u_{2j+1}) \in F(\Gamma)$ adjacent to u_0 , we see that, according to the cancellation property leading to the Toda systems, the product

$$\prod_{1 \leq j \leq n}^{\curvearrowright} U(u_0, u_{2j}; \alpha_j, \alpha_{j+1})$$

generates an identical Möbius transformation and is therefore a scalar matrix. Moreover, the latter property is equivalent to the discrete Toda equation at the vertex $u_0 \in V(\Gamma_0)$, formulated now in terms of the variables (u_0, u_{2j}) . Thus, we see that the following holds.

Proposition 3. *Discrete Toda equation (16) at the vertex $u_0 \in V(\Gamma_0)$ is rational in terms of u_0 and u_{2j} , $j = 1, \dots, n$. The discrete Toda system on an arbitrary graph Γ_0 admits a zero curvature representation in the sense of [5], with the transition matrices across the edge $(u_0, u_2) \in E(\Gamma_0)$ given by $U(u_0, u_2; \alpha_1 + \lambda, \alpha_2 + \lambda)$.*

The fact that the Toda equation for the star of the vertex u_0 is rational with respect to $u_{2j} = \wp(x_{2j})$, $j = 1, \dots, n$, is easy to see from (16), which can be rewritten with the help of the formula (6) as

$$\prod_{j=1}^n \frac{\wp(x_{2j}) - \wp(x_0 + \alpha_j - \alpha_{j+1})}{\wp(x_{2j}) - \wp(x_0 - \alpha_j + \alpha_{j+1})} \cdot \frac{\sigma^2(x_0 + \alpha_j - \alpha_{j+1})}{\sigma^2(x_0 - \alpha_j + \alpha_{j+1})} = 1.$$

However, the rationality with respect to $u_0 = \wp(x_0)$ remains hidden in this representation. This is a new manifestation of the fact that the original (polynomial in u_j) and the three-leg forms of the equation Q_4 are complementary in showing and hiding various properties.

The next, Lagrangian property is better visible again in the variables x , with the help of the three-leg form. It is easy to see that the function $\partial f(x_0, x_1; \alpha) / \partial x_1$ is symmetric with respect to the flip $x_0 \leftrightarrow x_1$. Integrating this function with respect to x_0 and x_1 , we see that there exists a symmetric function $\Lambda(x_0, x_1; \alpha) = \Lambda(x_1, x_0; \alpha)$, such that

$$f(x_0, x_1; \alpha) = \partial \Lambda(x_0, x_1; \alpha) / \partial x_0.$$

Proposition 4. *Let $E(\Gamma_0)$ be the set of the edges of the black graph Γ_0 . Let the pairs of parameters be assigned to the edges from $E(\Gamma_0)$ according to Fig. 5, so that, e.g., the pair (α_1, α_2) corresponds to the edge (x_0, x_2) . Then the discrete Toda equations (17) are Euler–Lagrange equations for the action functional*

$$S = \sum_{(x_0, x_2) \in E(\Gamma_0)} \Lambda(x_0, x_2; \alpha_1 - \alpha_2).$$

Finally, we give the matrix formulation for the cancellation phenomenon which is the main feature of the three-leg form.

Lemma 5. *If the relation*

$$f(x_0, x_1; \alpha) - f(x_0, x_3; \beta) = q \quad \Leftrightarrow \quad F(x_0, x_1; \alpha)/F(x_0, x_3; \beta) = e^{2q} \quad (19)$$

holds for some quantity q , then $u_3 = \wp(x_3)$ is a Möbius transformation of $u_1 = \wp(x_1)$ with the matrix

$$L = e^q N(x_0; \beta, \alpha) - e^{-q} N(x_0; -\beta, -\alpha), \quad (20)$$

where

$$N(x; \beta, \alpha) = \frac{\sigma^2(x + \beta)\sigma^2(x - \alpha)}{\sigma(2x)} \begin{pmatrix} -\wp(x + \beta) & \wp(x + \beta)\wp(x - \alpha) \\ -1 & \wp(x - \alpha) \end{pmatrix}. \quad (21)$$

Proof. According to (6), the formula (19) is equivalent to:

$$\frac{\wp(x_3) - \wp(x_0 + \beta)}{\wp(x_3) - \wp(x_0 - \beta)} \cdot \frac{\sigma^2(x_0 + \beta)}{\sigma^2(x_0 - \beta)} = \frac{\wp(x_1) - \wp(x_0 + \alpha)}{\wp(x_1) - \wp(x_0 - \alpha)} \cdot \frac{\sigma^2(x_0 + \alpha)}{\sigma^2(x_0 - \alpha)} \cdot e^{-2q}.$$

Now it is a matter of a straightforward computation to represent this as the Möbius transformation from $\wp(x_1)$ to $\wp(x_3)$ with the matrix L given by (20), (21). \square

Of course, matrix L is defined only up to a scalar factor, which is chosen in (20), (21) so that $\det L = \sigma(2\alpha)\sigma(2\beta) = \text{const.}$ The telescopic cancellation taking place by adding several equations of the type (19), is translated for the matrices L into the following property: if, additionally to (20), we have

$$L_1 = e^{q_1} N(x_0; \gamma, \beta) - e^{-q_1} N(x_0; -\gamma, -\beta),$$

then

$$L_1 L = \sigma(2\beta) (e^{q_1+q} N(x_0; \gamma, \alpha) - e^{-q_1-q} N(x_0; -\gamma, -\alpha)).$$

This follows easily from the properties of the rank 1 matrices N , namely:

$$N(x; \gamma, \beta) N(x; \beta, \alpha) = \sigma(2\beta) N(x; \gamma, \alpha), \quad N(x; \gamma, \beta) N(x; -\beta, -\alpha) = 0.$$

Notice that the case $q = f(x_0, x_2; \alpha - \beta)$ in eq. (19) corresponds to the additive three-leg form (4) of equation Q_4 . In this case we obtain from Lemma 5 a new formula for the transition matrices from Proposition 3:

$$U(u_0, u_2; \alpha_1, \alpha_2) \sim e^{f(x_0, x_2; \alpha_1 - \alpha_2)} N(x_0; \alpha_2, \alpha_1) - e^{-f(x_0, x_2; \alpha_1 - \alpha_2)} N(x_0; -\alpha_2, -\alpha_1),$$

where symbol \sim means equality up to a constant scalar factor. A direct verification of this formula is actually equivalent to a proof of Proposition 1.

4 Lagrangian systems with discrete time and space-time slicing of regular lattices

Now we consider the specification of the general scheme corresponding to the case of the shift-invariant graphs, or lattices. Recall the general construction of the discrete time Lagrangian mechanics [16]. Let \mathcal{X} be a vector space (the theory can be developed also in a more general setting, when \mathcal{X} is a manifold, but we will not need this here). Let $\mathcal{L} : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ be a smooth function, called the *discrete time Lagrange function*. For an arbitrary sequence $X : \varepsilon\mathbb{Z} \supset \{a\varepsilon, a\varepsilon + \varepsilon, \dots, b\varepsilon\} \mapsto \mathcal{X}$ one considers the *discrete time action functional*

$$S = \sum_{n=a}^{b-1} \mathcal{L}(X(n\varepsilon), X(n\varepsilon + \varepsilon)).$$

The sequences serving as critical points of the action functional (in the class of variations preserving $X(a\varepsilon)$ and $X(b\varepsilon)$) satisfy the *discrete time Euler-Lagrange equation*, which we shall write in the index-free form as

$$\partial\mathcal{L}(X, \tilde{X})/\partial X + \partial\mathcal{L}(\underline{X}, X)/\partial X = 0 \quad (22)$$

where \underline{X} , X , \tilde{X} stand for $X(n\varepsilon - \varepsilon)$, $X(n\varepsilon)$, $X(n\varepsilon + \varepsilon)$, respectively. This is an implicit equation for \tilde{X} . In general, it has more than one solution, and therefore defines a correspondence (multi-valued map) $(\underline{X}, X) \mapsto (X, \tilde{X})$. To discuss symplectic properties of this correspondence, define

$$P = \partial\mathcal{L}(\underline{X}, X)/\partial X \in T_X^*\mathcal{X}.$$

Then (22) may be rewritten as the system

$$P = -\partial\mathcal{L}(X, \tilde{X})/\partial X, \quad \tilde{P} = \partial\mathcal{L}(X, \tilde{X})/\partial \tilde{X}. \quad (23)$$

This system will be called the *Hamiltonian form* of eq. (22). It defines a multi-valued map $(X, P) \mapsto (\tilde{X}, \tilde{P})$ on $T^*\mathcal{X}$. More precisely, the first equation in (23) is an implicit equation for \tilde{X} , while the second one allows one to calculate \tilde{P} explicitly and uniquely, once X and \tilde{X} are known. The fundamental property of this multi-valued map is the following: *each branch of the map $T^*\mathcal{X} \mapsto T^*\mathcal{X}$ defined by (23) is symplectic with respect to the standard symplectic structure on $T^*\mathcal{X}$.*

The continuous limit of this construction is as follows. Suppose that $X(n\varepsilon)$ approximates the smooth function $X(t)$ at $t = n\varepsilon$ with some small $\varepsilon > 0$. Suppose that by the substitution $(X, \tilde{X}) = (X, X + \varepsilon\dot{X} + o(\varepsilon))$ there holds an asymptotic relation:

$$\mathcal{L}(X, \tilde{X}) \approx \varepsilon L(X, \dot{X})$$

with some smooth function $L : T\mathcal{X} \mapsto \mathbb{R}$, where the sign \approx means equality up to additive terms $o(\varepsilon)$. Then we have:

$$\begin{aligned} \partial L / \partial \dot{X} &\approx \partial\mathcal{L}(X, \tilde{X}) / \partial \tilde{X} = \tilde{P}, \\ \partial L / \partial X &\approx \varepsilon^{-1} (\partial\mathcal{L}(X, \tilde{X}) / \partial X + \partial\mathcal{L}(X, \tilde{X}) / \partial \tilde{X}) = \varepsilon^{-1} (\tilde{P} - P). \end{aligned}$$

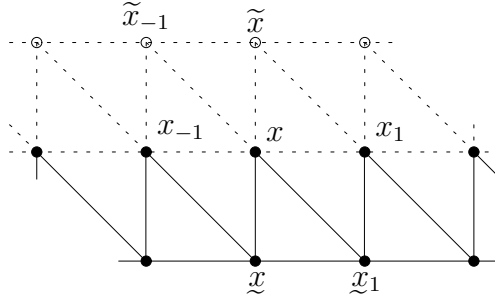


Fig. 6: Slice $(4+2)$ on triangular lattice

Thus, the discrete Lagrangian equations (23) approximate (serve as a discretization of) the continuous Euler–Lagrange equations with the Lagrangian $L(X, \dot{X})$:

$$P = \partial L(X, \dot{X}) / \partial \dot{X}, \quad \dot{P} = \partial L(X, \dot{X}) / \partial X.$$

Now we can introduce the discrete time Lagrangian formalism for discrete Toda systems on some regular two-dimensional lattices. We will consider planar graphs Γ with the set of vertices $V(\Gamma) = \mathbb{Z}^2$ only. They will be distinguished by their sets of edges $E(\Gamma)$. It will be convenient to think of the vertices $(k, n) \in \mathbb{Z}^2$ as representing the points $(k, n\varepsilon) \in \mathbb{R}^2$ with some (small) $\varepsilon > 0$. The first index k will enumerate the lattice sites, while the second index n has the meaning of the discrete time $t = n\varepsilon$. For a fixed $t = n\varepsilon$, we will write $V^t = \{(k, t) : k \in \mathbb{Z}\}$, so that $V(\Gamma) = \bigcup_{t \in \varepsilon\mathbb{Z}} V^t$.

A *slicing* of a graph Γ consists in choosing subgraphs Γ^t so that $V(\Gamma^t) = V^{t-\varepsilon} \cup V^t$, the sets $E(\Gamma^t) = E^t$ are disjoint, and $E(\Gamma) = \bigcup_{t \in \varepsilon\mathbb{Z}} E^t$.

The configuration space of our discrete time systems is $\mathcal{X} = \{X : \mathbb{Z} \mapsto \mathbb{R}\}$ consisting of real-valued sequences $X = (x_k)_{k \in \mathbb{Z}}$. An element $P \in T_X^* \mathcal{X}$ is a sequence $P = (p_k)_{k \in \mathbb{Z}}$. Clearly, for any fixed $t = n\varepsilon$, the set of functions on V^t is identified with \mathcal{X} . Functions on $V(\Gamma^t)$ are naturally identified with pairs of functions (\underline{X}, X) on $V^{t-\varepsilon}$ and V^t , respectively, and therefore belong to $\mathcal{X} \times \mathcal{X}$. The discrete time Lagrange functions for our Toda systems are of the form

$$\mathcal{L}(\underline{X}, X) = \sum_{(x, x') \in E^t} \Lambda(x, x'; \alpha - \beta),$$

i.e. are obtained by summing up the elementary Lagrangians over the edges of one slice of the graph Γ .

The pictures of the slicing for the graphs playing the main roles in our subsequent presentation are given below. Starting from this point, we use the following abbreviated notation for the sequences like $(x_k)_{k \in \mathbb{Z}}$: we write $x, x_{\pm 1}$ for $x_k, x_{k \pm 1}$, respectively. This notation has not to be confused with indices used in the previous section.

The central role in what follows will play the discrete Toda system on the triangular lattice and on the square lattice. All vertices of the regular triangular lattice have valence 6, so that the corresponding Toda system is a 7-point scheme. Similarly, the Toda system on the regular square lattice is a 5-point scheme. The Hamiltonian properties of these systems depend essentially on the slicing chosen. For the triangular lattice we will choose the slicing on Fig. 6. The corresponding map $(\underline{X}, X) \mapsto (X, \tilde{X})$ is defined implicitly,

since each equation contains two fields \tilde{x} and \tilde{x}_{-1} from the time level $t = n\varepsilon + \varepsilon$). For the regular square lattice we will use two different slicings. The first one is shown on Fig. 7 in a somewhat unusual way (by a standard drawing the slice would run diagonally). It corresponds to an implicit discrete Toda system. Indeed, also in this case each equation contains two fields \tilde{x} and \tilde{x}_{-1} from the time level $t = n\varepsilon + \varepsilon$. Finally, the slicing shown on Fig. 8 represents an explicit discrete Toda system, since each equation contains only one updated field \tilde{x} . These two types of 5-point equations can be obtained from the 7-point one by reduction consisting in erasing some edges of the triangular lattice.

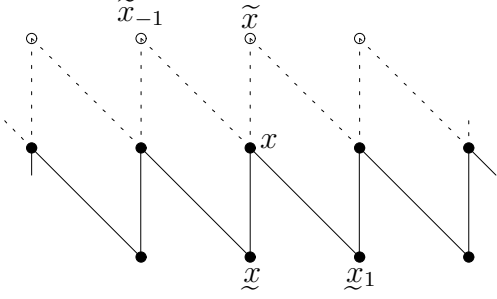


Fig. 7: Slice $(2 + 2)$ on square lattice

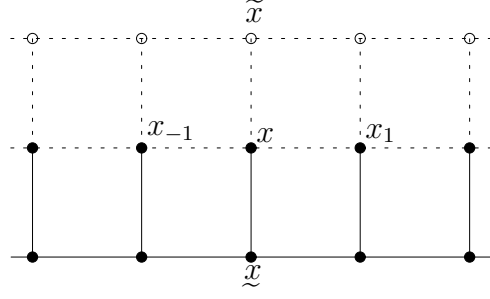


Fig. 8: Slice $(3 + 1)$ on square lattice

5 Triangular lattice

Time discretizations of lattices of the Ruijsenaars–Toda type [21] were introduced in [23, 24]. They were identified in [3] as discrete systems on the regular triangular lattice.

The regular triangular lattice T can be considered as the black subgraph of the quad-graph known as the *dual kagome lattice* (drawn on Fig. 9 in dashed lines). The latter graph has vertices of two kinds, black vertices of valence 6 and white vertices of valence 3, and edges of three types, all edges of each type being parallel. The parameters corresponding to these three types of edges are denoted $\alpha_1, \alpha_2, \alpha_3$ as shown on Fig. 9. It is easy to see that the parameters α_3 are constant along horizontal stripes, that is $\alpha_3 = \alpha_{3,n}$ and analogously, $\alpha_2 = \alpha_{2,k}$ and $\alpha_1 = \alpha_{1,k+n}$.

In what follows, we restrict ourselves to the particular case which admits a well-defined continuous limit, when the parameters α_1 and α_3 are constant:

$$\alpha_1 = \lambda, \quad \alpha_{2,k} = \lambda - \beta_k, \quad \alpha_3 = \lambda - \varepsilon, \quad (24)$$

where λ will play the role of the spectral parameter in the zero curvature representation of the Toda system. The discrete Lagrange function obtained by summing up the elementary Lagrangians along all edges of a time slice, is equal to

$$\mathcal{L}(\underline{X}, X) = \sum_k \left(\Lambda(x_k, \underline{x}_k; \varepsilon) + \Lambda(x_k, \underline{x}_{k+1}; \beta_k - \varepsilon) + \Lambda(\underline{x}_k, \underline{x}_{k+1}; -\beta_k) \right). \quad (25)$$

Equation of the *discrete elliptic Toda system on the triangular lattice* reads:

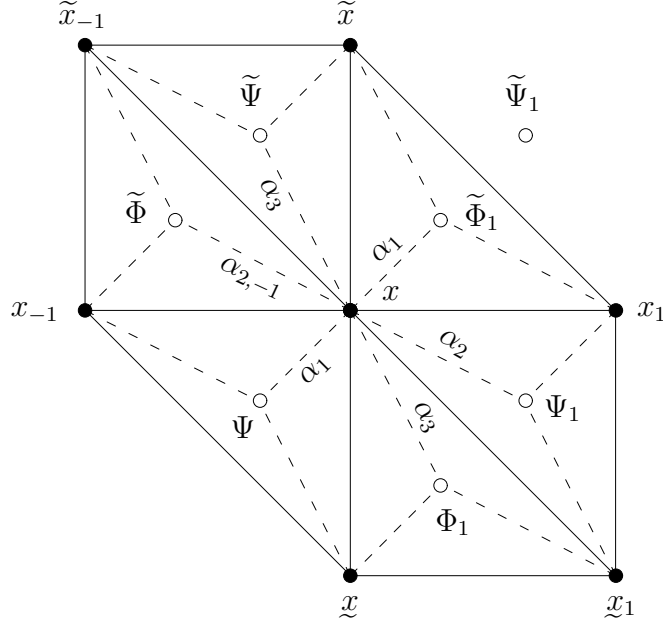


Fig. 9: Fields and wave functions on the triangular lattice

$$f(x, x_1; -\beta) + f(x, \tilde{x}; \varepsilon) + f(x, \tilde{x}_{-1}; \beta_{-1} - \varepsilon) + f(x, x_{-1}; -\beta_{-1}) + f(x, \underline{x}; \varepsilon) + f(x, \underline{x}_1; \beta - \varepsilon) = 0. \quad (26)$$

Using the fact that $f(x, x'; -\alpha) = -f(x, x'; \alpha)$, we put the *Hamiltonian form* of this equation as:

$$p = -f(x, \tilde{x}; \varepsilon) + f(x, x_1; \beta) + f(x, x_{-1}; \beta_{-1}) - f(x, \tilde{x}_{-1}; \beta_{-1} - \varepsilon), \quad (27)$$

$$\tilde{p} = f(\tilde{x}, x; \varepsilon) + f(\tilde{x}, x_1; \beta - \varepsilon). \quad (28)$$

Proposition 6. *The system (27), (28) admits a zero curvature representation*

$$\tilde{L}V = V_1L \quad (29)$$

with the matrices

$$L = L(x, p, \beta; \lambda) = e^p M(x, \beta; \lambda) + e^{-p} M(x, -\beta; -\lambda), \quad (30)$$

where the matrix $M(x, \beta; \lambda) = N(x; \lambda - \beta, \lambda) / \sigma(2\lambda)$ is given by

$$M(x, \beta; \lambda) = \frac{\sigma^2(x + \lambda - \beta)\sigma^2(x - \lambda)}{\sigma(2x)\sigma(2\lambda)} \begin{pmatrix} -\wp(x + \lambda - \beta) & \wp(x + \lambda - \beta)\wp(x - \lambda) \\ -1 & \wp(x - \lambda) \end{pmatrix}. \quad (31)$$

The matrices V in (29) are given by

$$V = V(x, x_{-1}, \tilde{x}_{-1}, \varepsilon; \lambda) = e^{\varepsilon G} M(x, \varepsilon; \lambda) + e^{-\varepsilon G} M(x, -\varepsilon; -\lambda), \quad (32)$$

where

$$\varepsilon G = f(x, x_{-1}; \beta_{-1}) - f(x, \tilde{x}_{-1}; \beta_{-1} - \varepsilon). \quad (33)$$

Proof. Consider the full system of the equations (1) on the dual kagome lattice and denote the vertices of the white sublattice as shown on the Fig. 9. The auxiliary linear problem $\Psi_1 = L\Psi$ corresponds to crossing two edges (x, \underline{x}) and (x, \underline{x}_1) . Adding the three-leg equations on the quadrilaterals $(x, \Psi, \underline{x}, \Phi_1)$ and $(x, \Phi_1, \underline{x}_1, \Psi_1)$, and using the formula (28) we find:

$$f(x, \Psi; \lambda) - f(x, \Psi_1; \lambda - \beta) = f(x, \underline{x}; \varepsilon) + f(x, \underline{x}_1; \beta - \varepsilon) = p. \quad (34)$$

Similarly, the auxiliary linear problem $\tilde{\Psi} = V\Psi$ corresponds to crossing two edges (x, x_{-1}) and (x, \tilde{x}_{-1}) . To find V , add the three-leg equations on two quadrilaterals $(x, \tilde{\Phi}, \tilde{x}_{-1}, \tilde{\Psi})$ and $(x, \Psi, x_{-1}, \tilde{\Phi})$:

$$f(x, \Psi; \lambda) - f(x, \tilde{\Psi}; \lambda - \varepsilon) = \varepsilon G, \quad (35)$$

with εG defined by (33). Now Lemma 5 applied to (34), (35), yields the formulas (30), (32), respectively. \square

Of course, in the Hamiltonian framework the main role in the zero curvature representation belongs to L , the transition matrix along the lattice. For instance, the so-called monodromy matrix $\prod_k L(x_k, p_k, \beta_k; \lambda)$ is a generating function of integrals of motion. In this context it is important that L_k is local, i.e. depends on one pair of canonically conjugate variables x_k, p_k only. As for the matrix V , it specifies just one (discrete) flow of the hierarchy. In the next section we will present another (continuous) commuting flow.

6 Elliptic Ruijsenaars–Toda lattice

The Lagrangian (25) has a well defined limit at $\varepsilon \rightarrow 0$, in the sense that $\mathcal{L}(X, X + \varepsilon \dot{X}) \approx \varepsilon L(X, \dot{X})$. Alternatively, this limit can be performed directly in the equations of motion (27), (28). Taking into account the relations

$$\lim_{\varepsilon \rightarrow 0} f(\tilde{x}, x; \varepsilon) = \log \frac{\dot{x} + 1}{\dot{x} - 1}, \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (f(\tilde{x}, x; \varepsilon) + f(x, \tilde{x}; \varepsilon)) = 2\zeta(2x),$$

one finds, as the limit of the equation (28) and the one obtained by subtracting (27) from (28):

$$p = \log \frac{\dot{x} + 1}{\dot{x} - 1} + f(x, x_1; \beta), \quad (36)$$

$$\dot{p} = \dot{x} \frac{\partial f(x, x_1; \beta)}{\partial x} + \dot{x}_{-1} \frac{\partial f(x, x_{-1}; \beta_{-1})}{\partial x_{-1}} - \frac{\partial f(x, x_1; \beta)}{\partial \beta} - \frac{\partial f(x, x_{-1}; \beta_{-1})}{\partial \beta_{-1}} + 2\zeta(2x). \quad (37)$$

From here the Newtonian equations of the *elliptic Ruijsenaars–Toda lattice* follow:

$$\begin{aligned} \frac{\ddot{x}}{\dot{x}^2 - 1} &= \dot{x}_1 \frac{\partial f(x, x_1; \beta)}{\partial x_1} - \dot{x}_{-1} \frac{\partial f(x, x_{-1}; \beta_{-1})}{\partial x_{-1}} \\ &\quad + \frac{\partial f(x, x_1; \beta)}{\partial \beta} + \frac{\partial f(x, x_{-1}; \beta_{-1})}{\partial \beta_{-1}} - 2\zeta(2x), \end{aligned} \quad (38)$$

where

$$\begin{aligned}\frac{\partial f(x, x_1; \beta)}{\partial x_1} &= \frac{1}{2}(\zeta(x + x_1 + \beta) - \zeta(x - x_1 + \beta) - \zeta(x + x_1 - \beta) + \zeta(x - x_1 - \beta)), \\ \frac{\partial f(x, x_1; \beta)}{\partial \beta} &= \frac{1}{2}(\zeta(x + x_1 + \beta) + \zeta(x - x_1 + \beta) + \zeta(x + x_1 - \beta) + \zeta(x - x_1 - \beta)).\end{aligned}$$

A remarkable feature of the zero curvature representation of Proposition 6 is that the matrix L in (30) does not depend on ε . Therefore the flow (36), (37) shares the matrix L with the map (27), (28). In other words, the latter is a Bäcklund transformation of the former.

Proposition 7. *Elliptic RTL (36), (37) admits a zero curvature representation*

$$\dot{L} = A_1 L - L A \quad (39)$$

with L from (30) and $A = A(x, x_{-1}, \dot{x}_{-1}; \lambda) = dV/d\varepsilon|_{\varepsilon=0}$, where V is given in (32).

By the usual substitution $u = \wp(x)$, the elliptic RTL (38) is brought into a rational form (which appeared first in [1]):

$$\frac{2\ddot{u} - r'(u)}{\dot{u}^2 - r(u)} = -\frac{\dot{u}_1}{h(u, u_1; \beta)} + \frac{\dot{u}_{-1}}{h(u, u_{-1}; \beta_{-1})} + \frac{\partial}{\partial u} \log(h(u, u_1; \beta)h(u, u_{-1}; \beta_{-1})), \quad (40)$$

where h is a biquadratic polynomial with discriminant $r(u)$ defined in Appendix B. By the derivation of eq. (40) from (38) the following alternative expression is used:

$$\begin{aligned}2\frac{\partial f(x, x_1; \beta)}{\partial x_1} &= \frac{\sigma(2\beta)\sigma(2x)\sigma(2x_1)}{\sigma(x + x_1 + \beta)\sigma(x - x_1 + \beta)\sigma(x + x_1 - \beta)\sigma(x - x_1 - \beta)} \\ &= -\frac{\wp'(x)\wp'(x_1)}{h(u, u_1; \beta)}.\end{aligned}$$

It is obtained with the help of (B.2), (B.6), (B.7). As a consequence, one finds:

$$\begin{aligned}2\frac{\partial f(x, x_1; \beta)}{\partial \beta} &= -\frac{\partial}{\partial x} \log \frac{\partial f(x, x_1; \beta)}{\partial x_1} + 2\zeta(2x) \\ &= \wp'(x) \frac{\partial}{\partial u} \log h(u, u_1; \beta) - \frac{\wp''(x)}{\wp'(x)} + 2\zeta(2x),\end{aligned}$$

which is the second expression used in the derivation of eq. (40).

7 Triangular lattice in \mathbb{Z}^3

Proposition 8. *Consider a solution $x : V(T) \mapsto \mathbb{C}$ of the discrete Toda system (26) on the triangular lattice T . Define a function $y : V(T) \mapsto \mathbb{C}$ by equations*

$$F(x, \underline{x}; \varepsilon) = F(x, y; \beta) / F(x, \underline{x}_1; \beta - \varepsilon). \quad (41)$$

Then also the following equations hold:

$$F(y, \tilde{y}; \varepsilon) = F(y, x; \beta) / F(y, \tilde{y}_{-1}; \beta - \varepsilon). \quad (42)$$

Moreover, y is also a solution of the discrete Toda system (26).

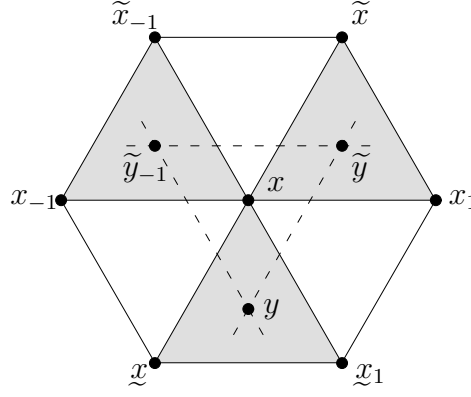


Fig. 10: Map $x \mapsto y$

The geometric description of equation (41) is the following: the variables y should be thought of as defined on the vertices of the new copy of triangular lattice which correspond to the half of the faces of the original one, namely those shaded on the Fig. 10 (so, y sits on vertex dual to the triangle $(x, \underline{x}, \underline{x}_1)$). Then equation (42) reflects the fact that the variables x are attached to the half of the faces of the y -lattice. Moreover, equations (41), (42) are nothing but the three-leg forms (8) of the basic equation Q_4 for the tetrahedra $(x, \underline{x}, \underline{x}_1, y)$ and $(y, \tilde{y}, \tilde{y}_{-1}, x)$. In terms of $u = \wp(x)$, $v = \wp(y)$, the equations (41), (42) read:

$$Q(u, \underline{u}, \underline{u}_1, v; \varepsilon, \beta) = 0, \quad Q(v, \tilde{v}, \tilde{v}_{-1}, u; \varepsilon, \beta) = 0.$$

Proof. Consider the system on the cubic lattice $\mathbb{Z}^3 = \{(i_1, i_2, i_3)\}$ consisting of equation Q_4 on each square face, with the parameters α_m attached to the edges of the direction i_m . This is possible due to the 3D consistency of equation Q_4 established in Proposition 2. Vertices of the x - and y -lattices can be interpreted as the points of the cubic lattice lying in the planes $i_1 + i_2 + i_3 = 0$ and $i_1 + i_2 + i_3 = 2$, respectively, see Fig. 11. Due to the three-leg property, both fields x and y satisfy the discrete Toda system (26). Equations (41), (42) are nothing but the tetrahedron equations (15) for the corresponding cubes.

Conversely, it is easy to see that, given a solution x of (26), one reconstructs uniquely the black sublattice of the cubic lattice, and then an arbitrary value at one white vertex determines the white sublattice uniquely. \square

So, equations (41), (42) can be considered as a sort of Bäcklund transformation $x \mapsto y$ of the discrete Toda system (26). On the other hand, they can be considered as a discrete time system (mapping) $(\underline{X}, Y) \mapsto (X, \tilde{Y})$, where $X = (x_k)_{k \in \mathbb{Z}}$, $Y = (y_k)_{k \in \mathbb{Z}}$. In this interpretation, this mapping serves as discretization of the Shabat-Yamilov lattice considered in the next section.

Proposition 9. *The discrete time system (41), (42) admits a zero curvature representation (29) with the matrices*

$$L(x, y, \beta; \lambda) = U(u, v; \lambda, \lambda - \beta), \quad (43)$$

$$V(x, \tilde{y}_{-1}, \varepsilon; \lambda) = U(u, \tilde{v}_{-1}; \lambda, \lambda - \varepsilon), \quad (44)$$

where $u = \wp(x)$, $v = \wp(y)$.

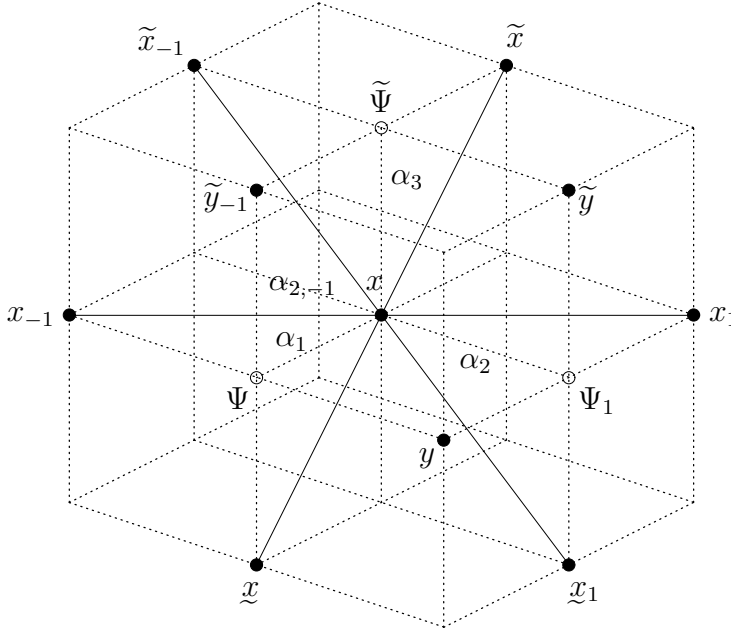


Fig. 11: Embedding of the triangular lattice into the cubic one

Proof. The matrices L , V correspond, as before, to the transitions $\Psi \rightarrow \Psi_1$ and $\Psi \rightarrow \tilde{\Psi}$. As can be seen from Fig. 11, these are transitions across diagonals of square faces (x, Ψ, y, Ψ_1) and $(x, \Psi, \tilde{y}_{-1}, \tilde{\Psi})$ of the cubic lattice. \square

8 Shabat-Yamilov lattice

The continuous limit $\varepsilon \rightarrow 0$ in equations (41), (42) leads to the differential-difference equations

$$\frac{\dot{x} + 1}{\dot{x} - 1} = \frac{F(x, y; \beta)}{F(x, x_1; \beta)}, \quad \frac{\dot{y} - 1}{\dot{y} + 1} = \frac{F(y, x; \beta)}{F(y, y_{-1}; \beta)}. \quad (45)$$

It should be noticed that elimination of y from these equations yields ERTL (38).

Under the change of variables $u = \wp(x)$, $v = \wp(y)$, the system (45) is brought into a rational form:

$$\dot{u} = \frac{2\rho(u_1, u, v; \beta)}{u_1 - v}, \quad \dot{v} = \frac{2\rho(u, v, v_{-1}; \beta)}{u - v_{-1}} \quad (46)$$

where the polynomial ρ is given in Appendix B. In the derivation of (46) from (45) one uses the formula

$$\frac{F(x, y; \beta)}{F(x, x_1; \beta)} = \frac{\wp(y) - \wp(x + \beta)}{\wp(y) - \wp(x - \beta)} \cdot \frac{\wp(x_1) - \wp(x - \beta)}{\wp(x_1) - \wp(x + \beta)},$$

and then takes (B.8) into account.

The system (46) was discovered in [22]. Its relation to the semi-discretization of the LL equation is discussed in [4].

Proposition 10. *The system (46) admits a zero curvature representation (39) with the matrices L from (43) and $A = A(u, v_{-1}; \lambda) = dU(u, v_{-1}; \lambda, \lambda - \varepsilon)/d\varepsilon|_{\varepsilon=0}$.*

9 Elliptic Volterra lattice

The same system on the cubic lattice that produced in Sect. 7 the discrete time Shabat-Yamilov lattice serves as the origin of another important lattice system and its time discretization. To show this, consider one row of elementary cubes and denote the fields variables as on Fig. 12. The parameters α_i take the same values (24) as before. Note that in these notations α_2 (and therefore β) are site-independent. Then it is easy to see that the fields x and \tilde{x} satisfy the equation

$$F(x, \tilde{x}; \varepsilon) = F(x, x_1; \beta) / F(x, \tilde{x}_{-1}; \beta - \varepsilon). \quad (47)$$

This equation is nothing but the three-leg form (8) of the basic equation Q_4 for the tetrahedron $(x, x_1, \tilde{x}, \tilde{x}_{-1})$. It defines an (implicit) map $X \mapsto \tilde{X}$, where, as usual, $X = (x_k)_{k \in \mathbb{Z}}$.

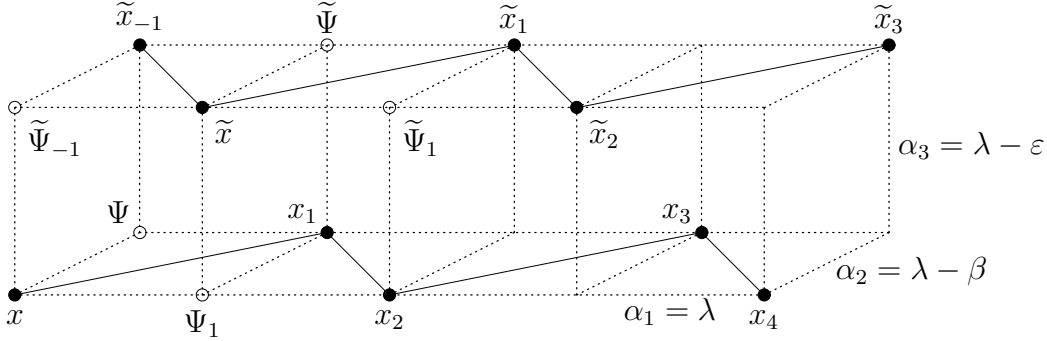


Fig. 12: Discrete time Volterra lattice

Proposition 11. *The map (47) admits a zero curvature representation (29) with the matrices*

$$\begin{aligned} L &= L(x_1, x, \beta; \lambda) = U(u_1, u; \lambda, \lambda - \beta), \\ V &= V(x_1, \tilde{x}_{-1}, \varepsilon; \lambda) = U(u_1, \tilde{u}_{-1}; \lambda, \lambda - \varepsilon). \end{aligned}$$

Proof. As usual, the matrix L corresponds to the transition $\Psi \rightarrow \Psi_1$, the matrix V corresponds to the transition $\Psi \rightarrow \tilde{\Psi}$. One sees readily from Fig. 12 that these are transitions across diagonals of elementary squares. \square

In the limit $\varepsilon \rightarrow 0$ the equation (47) turns into the *elliptic Volterra lattice*

$$\frac{\dot{x} - 1}{\dot{x} + 1} = \frac{F(x, x_1; \beta)}{F(x, x_{-1}; \beta)}. \quad (48)$$

In the rational form,

$$\dot{u} = \frac{2\rho(u_1, u, u_{-1}; \beta)}{u_1 - u_{-1}},$$

it is due to Yamilov [27]. This lattice admits a zero curvature representation (39) with the same matrix L as in the discrete time model. In particular, the map (47) is a Bäcklund transformation for the flow (48).

10 Square lattice, implicit scheme

We consider now the skew square lattice with a slicing as on Fig. 7, obtained from the triangular one by erasing the horizontal edges. This can be achieved by setting $\alpha_1 = \alpha_2$, so that for our particular setting (24) all parameters β_k of the Toda lattice vanish, and the parameters of the corresponding quad-graph (which is a finer square lattice shown by dashes on Fig. 13) are $\alpha_1 = \lambda$, $\alpha_3 = \lambda - \varepsilon$.

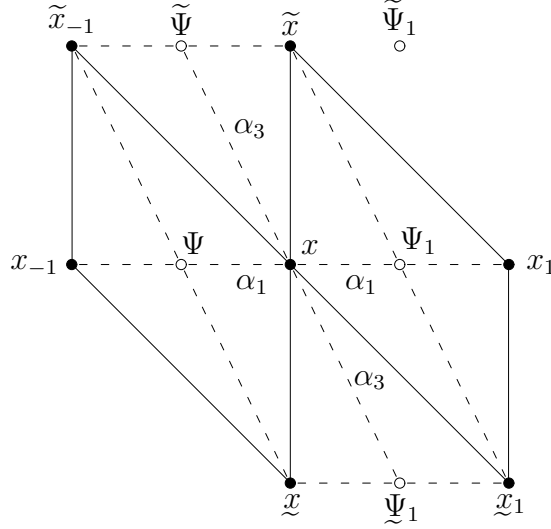


Fig. 13: Fields and wave functions on a skew square lattice

All results on the discrete Toda system on the skew square lattice are thus obtained from those of Sect.5 by setting $\beta = 0$. The discrete Lagrange function obtained by summing up the elementary Lagrangians along all edges of a time slice, is equal to

$$\mathcal{L}(\underline{X}, \underline{X}) = \sum_k (\Lambda(x_k, \underline{x}_k; \varepsilon) + \Lambda(x_k, \underline{x}_{k+1}; -\varepsilon)).$$

The equations of the Toda system read:

$$f(x, \tilde{x}; \varepsilon) + f(x, \tilde{x}_{-1}; -\varepsilon) + f(x, \underline{x}; \varepsilon) + f(x, \underline{x}_1; -\varepsilon) = 0. \quad (49)$$

The Hamiltonian form of these equations is

$$p = -f(x, \tilde{x}; \varepsilon) + f(x, \tilde{x}_{-1}; \varepsilon), \quad \tilde{p} = f(\tilde{x}, x; \varepsilon) - f(\tilde{x}, x_1; \varepsilon). \quad (50)$$

Proposition 12. *The system (50) admits a zero curvature representation (29) with the matrices*

$$L = L(x, p; \lambda) = e^p M(x, 0; \lambda) + e^{-p} M(x, 0; -\lambda), \quad (51)$$

$$V = V(x, \tilde{x}_{-1}, \varepsilon; \lambda) = U(u, \tilde{u}_{-1}; \lambda, \lambda - \varepsilon). \quad (52)$$

Rewriting (49) in the multiplicative form,

$$\frac{F(x, \underline{x}_1; \varepsilon)}{F(x, \underline{x}; \varepsilon)} = \frac{F(x, \tilde{x}; \varepsilon)}{F(x, \tilde{x}_{-1}; \varepsilon)},$$

and then performing transformations like those mentioned at the end of Sect. 8, we put the discrete Toda system in the rational form:

$$\frac{\rho(\underline{u}_1, u, \underline{u}; \varepsilon)}{\underline{u}_1 - \underline{u}} = \frac{\rho(\tilde{u}, u, \tilde{u}_{-1}; \varepsilon)}{\tilde{u} - \tilde{u}_{-1}}.$$

In this form it was given in [4].

11 Elliptic Toda lattice

Performing the limit $\varepsilon \rightarrow 0$ in eqs. (50) (or $\beta \rightarrow 0$ in eqs. (36), (37)), we find:

$$p = \log \frac{\dot{x} + 1}{\dot{x} - 1}, \tag{53}$$

$$\dot{p} = -\zeta(x + x_1) + \zeta(x_1 - x) - \zeta(x + x_{-1}) - \zeta(x - x_{-1}) + 2\zeta(2x). \tag{54}$$

From (53), (54) there follow the Newtonian equations of motion:

$$\frac{\ddot{x}}{\dot{x}^2 - 1} = \zeta(x_1 + x) - \zeta(x_1 - x) + \zeta(x + x_{-1}) + \zeta(x - x_{-1}) - 2\zeta(2x).$$

This is the *elliptic Toda lattice* as given in [12]. The rational form of this equation (with $u = \wp(x)$, as usual) was discovered earlier in [22, 28] and reads:

$$\frac{\ddot{u} - r'(u)/2}{\dot{u}^2 - r(u)} = \frac{1}{u - u_1} + \frac{1}{u - u_{-1}}.$$

The matrix L the zero curvature representation of Proposition 12 does not depend on ε . As a consequence, this matrix serves also for the continuous time system.

Proposition 13. *The system (53), (54) admits a zero curvature representation (39) with the matrices L from (51) and $A = A(u, u_{-1}; \lambda) = dU(u, u_{-1}; \lambda, \lambda - \varepsilon)/d\varepsilon|_{\varepsilon=0}$.*

An alternative zero curvature representation was given in [12] (with a less local matrix L depending on x, p and x_1). The relation of both representation remains unclear.

12 Square lattice, explicit scheme

Last, we consider the square lattice with a slicing as on Fig. 8, obtained from the triangular one by erasing the diagonal edges. This can be achieved by setting $\alpha_2 = \alpha_3$. For our particular setting (24) this means that all parameters of the Toda lattice are equal: $\beta_k = \varepsilon$. The parameters of the corresponding quad-graph (finer square lattice shown by dashes on Fig. 14) are $\alpha_1 = \lambda$, $\alpha_2 = \lambda - \varepsilon$.

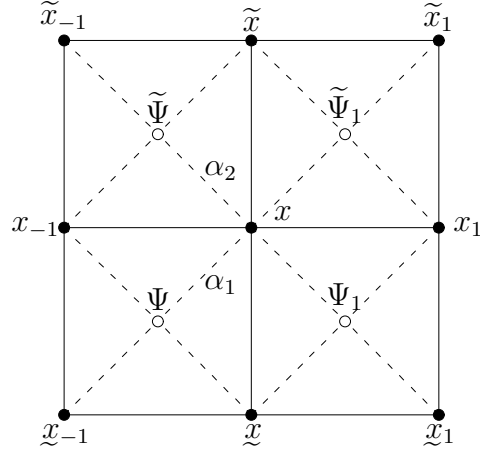


Fig. 14: Fields and wave functions on a square lattice

Correspondingly, the results on the discrete Toda system on the square lattice are obtained from the results of Sect. 5 by putting $\beta = \varepsilon$. The discrete Lagrange function is equal to

$$\mathcal{L}(X, X) = \sum_k (\Lambda(x_k, \underline{x}_k; \varepsilon) + \Lambda(\underline{x}_k, \underline{x}_{k+1}; -\varepsilon)).$$

The equations of the discrete Toda system read:

$$f(x, x_1; -\varepsilon) + f(x, \tilde{x}; \varepsilon) + f(x, x_{-1}; -\varepsilon) + f(x, \underline{x}; \varepsilon) = 0, \quad (55)$$

and their Hamiltonian form is

$$p = -f(x, \tilde{x}; \varepsilon) + f(x, x_1; \varepsilon) + f(x, x_{-1}; \varepsilon), \quad \tilde{p} = f(\tilde{x}, x; \varepsilon). \quad (56)$$

The rational form of (55) reads:

$$\frac{\rho(u_1, u, \underline{u}; \varepsilon)}{u_1 - \underline{u}} = \frac{\rho(\tilde{u}, u, u_{-1}; \varepsilon)}{\tilde{u} - u_{-1}}.$$

The zero curvature representation is of the form (29), with

$$L = L(x, p, \lambda) = e^p M(x, \varepsilon; \lambda) + e^{-p} M(x, -\varepsilon; -\lambda), \\ V = V(x, x_{-1}; \lambda) = U(u, u_{-1}; \lambda, \lambda - \varepsilon).$$

Notice also that, in virtue of (56), $p = f(x, \underline{x}; \varepsilon)$, hence $L = U(u, \underline{u}; \lambda, \lambda - \varepsilon)$.

It is important to mention that both maps (50) and (56) approximate in the continuous limit $\varepsilon \rightarrow 0$ the elliptic Toda lattice (53), (54). The first (implicit) map is a Bäcklund transformation of ETL itself (it shares the integrals of motion and commutes with ETL). However, the second (explicit) map is a Bäcklund transformation of ERTL with the parameters $\beta = \varepsilon$.

A List Q of integrable equations on quad-graphs

The first three items of the list Q_1 – Q_4 are

$$a(u_0 - u_3)(u_1 - u_2) - b(u_0 - u_1)(u_3 - u_2) + \delta^2 ab(a - b) = 0, \quad (Q_1)$$

$$a(u_0 - u_3)(u_1 - u_2) - b(u_0 - u_1)(u_3 - u_2) + ab(a - b)(u_0 + u_1 + u_2 + u_3) - ab(a - b)(a^2 - ab + b^2) = 0, \quad (Q_2)$$

$$b(a^2 - 1)(u_0 u_1 + u_2 u_3) - a(b^2 - 1)(u_0 u_3 + u_1 u_2) + (b^2 - a^2)(u_0 u_2 + u_1 u_3) - \delta^2(a^2 - b^2)(a^2 - 1)(b^2 - 1)/(4ab) = 0. \quad (Q_3)$$

The additive three-leg forms are collected in the table below. (In all cases except for $(Q_1)_{\delta=0}$ it is actually more convenient to deal with the multiplicative three-leg form (2).)

	$f(x_0, x_1; \alpha)$	$u = \phi(x)$	$a = \rho(\alpha)$
$(Q_1)_{\delta=0}$	$\frac{\alpha}{x_0 - x_1}$	x	α
$(Q_1)_{\delta=1}$	$\frac{1}{2} \log \frac{x_0 - x_1 + \alpha}{x_0 - x_1 - \alpha}$	x	α
$(Q_3)_{\delta=0}$	$\frac{1}{2} \log \frac{\sinh(x_0 - x_1 + \alpha)}{\sinh(x_0 - x_1 - \alpha)}$	$\exp 2x$	$\exp 2\alpha$
(Q_2)	$\frac{1}{2} \log \frac{(x_0 + x_1 + \alpha)(x_0 - x_1 + \alpha)}{(x_0 + x_1 - \alpha)(x_0 - x_1 - \alpha)}$	x^2	α
$(Q_3)_{\delta=1}$	$\frac{1}{2} \log \frac{\sinh(x_0 + x_1 + \alpha) \sinh(x_0 - x_1 + \alpha)}{\sinh(x_0 + x_1 - \alpha) \sinh(x_0 - x_1 - \alpha)}$	$\cosh 2x$	$\exp 2\alpha$
(Q_4)	$\frac{1}{2} \log \frac{\sigma(x_0 + x_1 + \alpha) \sigma(x_0 - x_1 + \alpha)}{\sigma(x_0 + x_1 - \alpha) \sigma(x_0 - x_1 - \alpha)}$	$\wp(x)$	$\wp(\alpha)$

According to the classification routine developed in [6], the main characteristic of an integrable equation (1) is a certain polynomial $r(u)$. For the equations Q_1 – Q_4 this polynomial is equal to

$$1, \quad u, \quad u^2 - \delta^2, \quad 4u^3 - g_2 u - g_3,$$

correspondingly. Using this information, one easily finds the following sequence of degenerations:

$$\begin{array}{ccccc}
 & & (Q_3)_{\delta=1} & \longrightarrow & (Q_3)_{\delta=0} \\
 & \nearrow & \downarrow & & \downarrow \\
 (Q_4) & & & & \\
 & \searrow & (Q_2) & \longrightarrow & (Q_1)_{\delta=1} \longrightarrow (Q_1)_{\delta=0}
 \end{array}$$

where

$$\begin{aligned}
(Q_4) \rightarrow (Q_3)_{\delta=1} : \quad & g_2 = \frac{4}{3}, \quad g_3 = -\frac{8}{27}, \\
& u \rightarrow \frac{2}{u-1} + \frac{1}{3}, \quad a \rightarrow \frac{4a}{(a-1)^2} + \frac{1}{3}, \quad A \rightarrow -\frac{8a(a+1)}{(a-1)^3}, \\
(Q_4) \rightarrow (Q_2) : \quad & g_2 = g_3 = 0, \quad u \rightarrow 1/u, \quad a \rightarrow a^{-2}, \quad A \rightarrow -2a^{-3}, \\
(Q_3)_{\delta=1} \rightarrow (Q_2) : \quad & u \rightarrow 1 + 2\epsilon^2 u, \quad a \rightarrow 1 + 2\epsilon a, \quad \epsilon \rightarrow 0, \\
(Q_3)_{\delta=1} \rightarrow (Q_3)_{\delta=0} : \quad & u \rightarrow u/\epsilon, \quad \epsilon \rightarrow 0, \\
(Q_2) \rightarrow (Q_1)_{\delta=1} : \quad & a \rightarrow \epsilon a, \quad u \rightarrow \frac{1}{4} + \epsilon u, \quad \epsilon \rightarrow 0, \\
(Q_3)_{\delta=0} \rightarrow (Q_1)_{\delta=1} : \quad & u \rightarrow 1 + 2\epsilon u, \quad a \rightarrow 1 + 2\epsilon a, \quad \epsilon \rightarrow 0, \\
(Q_1)_{\delta=1} \rightarrow (Q_1)_{\delta=0} : \quad & a \rightarrow \epsilon a, \quad \epsilon \rightarrow 0.
\end{aligned}$$

B Some formulas with elliptic functions

Some of the most useful formulas for the Weierstrass elliptic functions are:

$$\begin{aligned}
\sigma(x+\alpha)\sigma(x-\alpha)\sigma(\beta+\gamma)\sigma(\beta-\gamma) &= \sigma(x+\beta)\sigma(x-\beta)\sigma(\alpha+\gamma)\sigma(\alpha-\gamma) \\
&\quad - \sigma(x+\gamma)\sigma(x-\gamma)\sigma(\alpha+\beta)\sigma(\alpha-\beta), \quad (B.1)
\end{aligned}$$

$$\zeta(x) + \zeta(y) + \zeta(z) - \zeta(x+y+z) = \frac{\sigma(x+y)\sigma(y+z)\sigma(z+x)}{\sigma(x)\sigma(y)\sigma(z)\sigma(x+y+z)}, \quad (B.2)$$

$$\frac{1}{2} \begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(y) & \wp'(y) \\ 1 & \wp(z) & \wp'(z) \end{vmatrix} = \frac{\sigma(x+y+z)\sigma(x-y)\sigma(y-z)\sigma(z-x)}{\sigma^3(x)\sigma^3(y)\sigma^3(z)}, \quad (B.3)$$

$$\wp(x) - \wp(y) = -\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)}. \quad (B.4)$$

A fundamental object related to the elliptic curve \mathcal{E} is the polynomial

$$H(u, v, w) = (uv + vw + wu + g_2/4)^2 - (u + v + w)(4uvw - g_3). \quad (B.5)$$

Indeed, the equation $H(\wp(x), \wp(y), \wp(x \pm y)) = 0$ is the addition theorem for the \wp -function. This follows from the formula

$$H(\wp(x), \wp(y), \wp(z)) = -\frac{\sigma(x+y+z)\sigma(-x+y+z)\sigma(x-y+z)\sigma(x+y-z)}{\sigma^4(x)\sigma^4(y)\sigma^4(z)}. \quad (B.6)$$

There holds the identity $H_v^2 - 2HH_{vv} = r(u)r(w)$. This gives rise to a one-parameter family of symmetric biquadratics $h(u, v; \beta)$ with the property $h_v^2 - 2hh_{vv} = r(u)$:

$$h(u, v; \beta) = H(u, v, b)/\sqrt{r(b)} = H(u, v, \wp(\beta))/\wp'(\beta). \quad (B.7)$$

Recall [6] that the polynomial h is related to the basic polynomial Q from eq. (1) by the formula $Q_{u_2}Q_{u_3} - QQ_{u_2u_3} = k(\alpha, \beta)h(u_0, u_1; \alpha)$.

To an arbitrary quadratic polynomial $h(u) = c_2 u^2 + 2c_1 u + c_0$ there corresponds the symmetric affine-linear polynomial $\rho(u, w)$ with the property $h(u) = \rho(u, u)$, namely $\rho = c_2 u w + c_1(u + w) + c_0$. In a more invariant fashion, $2\rho = 2h - (u - w)h_u$. For the polynomial $h = h(u, v; \beta)$ from (B.7) this polynomial ρ is given by:

$$\rho(u, v, w; \beta) = \frac{(\wp(y) - \wp(\beta))^2}{\wp'(\beta)} \left(uw - (\wp(y + \beta) + \wp(y - \beta)) \frac{u + w}{2} + \wp(y + \beta)\wp(y - \beta) \right), \quad (\text{B.8})$$

where, as usual, $u = \wp(x)$, $v = \wp(y)$ and $w = \wp(z)$. This formula follows easily from

$$h(u, v; \beta) = \frac{(\wp(y) - \wp(\beta))^2}{\wp'(\beta)} (u - \wp(y + \beta))(u - \wp(y - \beta)),$$

which, in turn, is a direct consequence of (B.6), (B.7).

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